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THE ROLE OF $P_s$, $P_r$, AND $P_l$ IN CONSTITUTIVE EQUATION,
AND BOUNDARY CONDITION IN CAKE FILTRATION

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ABSTRACT
The constitutive equations proposed by Tiller and Shirato were analyzed and a new constitutive equation originated from the thickness of sediment was suggested. New boundary condition of a filter cake based on the solid compressive pressure of the first solid layer, $p_s$, was also proposed. With the new constitutive equation and boundary conditions, the average specific cake resistances at various pressures were calculated.

INTRODUCTION
The modern cake filtration theory was started with the Ruth's compression-permeability cell (CPC)[1]. Grace[2] performed many CPC experiments for various kind of particles, and measured the specific cake resistances with varying the solid compressive pressure, $p_s$. Tiller[3] proposed that the cake porosity($e$) and resistance($\alpha$) is constant below a certain solid compressive pressure $p_s$. Shirato[4] proposed a set of equations which could express the Tiller's notion concisely with $p_s$. Then Tiller and Crump[5] accepted it. In this study we analyze the notion of Tiller and Shirato, and want to find the true meaning of the necessity of those notions in cake filtration. With the sedimentation results which are essentially performed at a very low solid compressive pressure, the correctness of those notions was examined.

ANALYSIS OF THE CONSTITUTIVE EQUATIONS

Constitutive Equations of Tiller and Shirato

Tiller[3] proposed constitutive equations as below based on CPC experiments.

When $p_s$ is greater than $p_r$;

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\( \alpha = a p,^n \quad 1 - \varepsilon = B p,^p \) 

[1]

When \( p_s \) is smaller than \( p_i \):

\( \alpha = \alpha_i = a p,^n = \text{constant} \quad 1 - \varepsilon = 1 - \varepsilon_i = B p,^p = \text{constant} \) 

[2]

The Eqs. mean that the specific cake resistance and porosity do not change below a certain value of solid compressure \( p_i \). Tiller[5] demonstrated an experimental CPC result which shows \( p_i \) is about 1.4 kPa. Eqs. (1) and (2) shows a sharp change in specific cake resistance and porosity at \( p_i \) and it seems unnatural. Shirato[4] proposed a set of Eqs. which can express the above notion somewhat smoothly for the whole range of \( p_s \) and Tiller and Crump[5] changed the form of Eqs. as below. These Eqs. also cannot explain constant value of \( \alpha \) and \( \varepsilon \) below a certain solid compressive pressure.

\[ \frac{\alpha}{\alpha_s} = \left(1 + \frac{p_s}{p_a}\right)^n \quad \frac{(1 - \varepsilon)/(1 - \varepsilon_a)}{} = \left(1 + \frac{p_s}{p_a}\right)^p \] 

[3]

**Porosity at Low Compressive Pressure; Ylm’s Constitutive Eqs.**

Shirato et al.[6] measured the final height of sediment and proved that the porosity changes down to 100 Pa with Mitsukuri-gairome clay which has the compressibility of 0.7. In our experiments bentonite floc, compressibility of 1.125, changed the porosity down to 10 Pa. With these experimental evidences, we adopted that Eq. (1) is valid until a very small solid compressive pressure, i.e. \( p_s > 0 \). The experimental CPC data with bentonite floc and three constitutive Eqs. are shown in Fig. 1.

Fig. 1. Experimental results and constitutive equations proposed by Tiller, Shirato, and Ylm.

**NEW BOUNDARY CONDITION IN CAKE FILTRATION**

Until now, the solid compressive pressure at the first solid layer of a cake is considered as
zero. Solid compressive pressure is originated from the friction between particles and fluid. It was assumed that the solid compressive pressure of the first solid layer could not result in the variation of porosity as Eq. (1). The second layer receives the solid compressive pressure, \( p_n \), originated from the frictional force of the first solid layer. This solid compressive pressure results in the variation of porosity as Eq. (1). Thus, we propose the boundary conditions of a cake as Eq. (4). Here, \( p_s \) is solid compressive pressure of the first solid layer.

\[
\begin{align*}
  p_{s,up} = p_f &\neq 0 \\
  p_{s,down} = \Delta p_{cake} &
\end{align*}
\]

AVERAGE SPECIFIC CAKE RESISTANCES BY CONSTITUTIVE EQUATIONS

Average specific cake resistance induced by Eq. (1) is;

\[
\alpha_{ev} = \frac{\Delta p_e}{\alpha p_e} = \frac{\Delta p_e}{p_e} = \frac{a(1-n)\Delta p_e}{\Delta p_e^{1-n}} = a(1-n)\Delta p_e^{1-n}
\]

[5]

For a very compressible cake which has the compressibility greater than one, the average specific cake resistance by Eq. (5) become negative value and it cannot be happen. Eq. (6) is the average specific cake resistance with the Tiller's notion, i.e. from Eq. (1) and Eq. (2).

\[
\alpha_{ev} = \frac{\Delta p_e}{\alpha p_e} = \frac{\Delta p_e}{p_e} + \frac{\Delta p_e}{\alpha p_e} = \frac{a(1-n)\Delta p_e}{\Delta p_e^{1-n} - n p_f^{1-n}}
\]

[6]

The value of \((1 - n)\) has negative value also when \( n \) is greater than one, but the denominator also has the negative value in this case. Thus the average specific cake resistance is always positive when \( n \) is smaller or greater than unity. The author thinks that this is the true role of \( p_s \). Eq. (7) is the average specific cake resistance induced by the Shirato's suggestion, Eq. (3).

\[
\alpha_{ev} = \frac{\Delta p_e}{\alpha p_e} = \frac{\alpha_s (1-n)\Delta p_e}{\alpha_s (1 + p_f / p_e)^s \left( 1 + \frac{\Delta p_e}{p_e} \right)^{1-s} - 1}
\]

[7]

Eq. (7) also gives positive average specific cake resistance for the all range of \( n \). The average specific cake resistance induced by our hypothesis Eq. (1) and new boundary condition Eq. (4) is;

\[
\alpha_{ev} = \frac{\Delta p_e - p_f}{\alpha p_e} = \frac{\Delta p_e - p_f}{\alpha p_e} = \frac{a(1-n)(\Delta p_e - p_f)}{(\Delta p_e - p_f)^{1-s} - p_f^{1-s}} = \frac{a(1-n)\Delta p_e}{\Delta p_e^{1-n} - p_f^{1-n}}
\]

[8]
Different from Eq. (5), average specific cake resistance by Eq. (8) is always positive. The calculated and experimental average specific cake resistances with bentonite floc are shown in Fig. 2.

![Graph showing average specific cake resistances by experiments and various constitutive equations.]

**Fig. 2.** Average specific cake resistances by experiments and by various constitutive equations.

Each equation for calculating average specific cake resistance has different forms, and we proposed totally different notion for cake constitutive equation and boundary condition, the calculated results shows almost the same values and these results coincide well with experimental data. With our notion, the cake thickness is a function of the solid compressive pressure of the first solid layer, $p_r$. When $p_r$ increases by any other factors, the cake thickness decreases. This effect is prominent for the very compressible cake, i.e. compressibility greater than 1.

**CONCLUSION**

A new boundary condition and constitutive equation of filter cake has been proposed and the meaning is discussed.

**REFERENCES**